CS 784: Computational Linguistics - Waitlist Quiz

Full Name: _____ Student Number: _____

Problem 1 (25 pts) There is a bag with 3 red balls and 2 blue balls. All balls have the same probability of being drawn. Please beiefly explain your answer for each case.

1. (5 pts) We randomly draw a ball from the bag. What is the probability that the ball is red?

   $P(red) = \frac{\#\ red\ balls}{\#\ red\ balls + \#\ blue\ balls} = \frac{3}{3+2} = \frac{3}{5}$

2. (10 pts) We randomly draw a ball from the bag. Without putting it back, we randomly draw another ball from the bag. What is the probability that the second ball is red, if the first ball is red?

   $P(second = red \mid first = red) = \frac{\#\ red\ balls - 1}{\#\ red\ balls + \#\ blue\ balls - 1} = \frac{2}{3+2-1} = \frac{1}{2}$

3. (10 pts) We randomly draw a ball from the bag. Without putting it back, we randomly draw another ball from the bag. What is the probability that the second ball is red, no matter what the color of the first ball is?

$$\begin{aligned} P(second = red) =& P(second = red \mid first = red) \cdot P(first = red) + P(second = red \mid first = blue) \cdot P(first = blue) \\ =& \frac{1}{2} \cdot \frac{3}{5} + \frac{3}{4} \cdot \frac{2}{5} = \frac{3}{10} + \frac{3}{10} = \frac{3}{5} \end{aligned}$$

Problem 2 (30 pts) We have a function $f(x)$. For the following cases, what value of x minimizes $f(x)$ in the given domain? What is the minimum value of $f(x)$? Please briefly explain your answer.

1. (15 pts) $f(x) = x^2 - 2x + 1$, $x \in \mathbb{R}$

   Let $\frac{\mathrm{d}f(x)}{\mathrm{d}x} = 0$, we have $2x - 2 = 0 \Rightarrow x = 1$, so the minimum value is $f(1) = 0$.

2. (15 pts) $f(x) = x \log x - 5x + 4$, $x \in \mathbb{R}^+$

   Let $\frac{\mathrm{d}f(x)}{\mathrm{d}x} = 0$, we have $\log x + 1 - 5 = 0 \Rightarrow x = e^4$, so the minimum value is $f(e^4) = 4e^4 - 5e^4 + 4 = -e^4 + 4$.
   Note: $\mathrm{d}x \log x = \log x + \frac{1}{x} \cdot x = \log x + 1$, applying the rule of $(fg)' = f'g + fg'$.

Problem 3 (25 pts) Given the matrix: $\boldsymbol{A} = \begin{bmatrix} 3 & 2 & -2 \\ 1 & 3 & -1 \\ 1 & 2 & 0 \end{bmatrix}$

1. (10 pts) Write down a vector $\mathbf{x}$ such that $\boldsymbol{A}\mathbf{x} = \mathbf{x}$.

   $\mathbf{x} = [1, 0, 1]^\intercal$, $[k, 0, k]^\intercal$ and $[0, 0, 0]^\intercal$ also get full marks.

2. (15 pts) Find all eigenvalues of $\boldsymbol{A}$. Note: for an eigenvector $\mathbf{x}$ with eigenvalue λ, $\boldsymbol{A}\mathbf{x} = \lambda\mathbf{x}$.

The characteristic equation is $|\boldsymbol{A} - \lambda I| = 0$.
The determinant of $\boldsymbol{A} - \lambda \boldsymbol{I}$ is:

$$
\begin{aligned}
&|\boldsymbol{A} - \lambda I| \\
=&\begin{vmatrix} 3-\lambda & 2 & -2 \\ 1 & 3-\lambda & -1 \\ 1 & 2 & -\lambda \end{vmatrix} \\
=&(3-\lambda)\begin{vmatrix} 3-\lambda & -1 \\ 2 & -\lambda \end{vmatrix} - 2\begin{vmatrix} 1 & -1 \\ 1 & -\lambda \end{vmatrix} - 2\begin{vmatrix} 1 & 3-\lambda \\ 1 & 2 \end{vmatrix} \\
=&(3-\lambda)\left((3-\lambda)\cdot(-\lambda) - (-1)\cdot 2\right) - 2\left(1\cdot -\lambda - (-1)\cdot 1\right) - 2(1\cdot 2 - (3-\lambda)) \\
=&(3-\lambda)(\lambda^2 - 3\lambda + 2) + 2(\lambda - 1) - 2(\lambda - 1) \\
=&(3-\lambda)(\lambda^2 - 3\lambda + 2) = 0 \\
\Rightarrow&(3-\lambda)(\lambda - 1)(\lambda - 2) = 0
\end{aligned}
$$

Finally, we have $\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 3$.

Problem 4 (20 pts) What is the time complexity of this algorithm, in terms of n? Explain your answer.

```
def foo(a: list[int]):
   n = len(a)
   b = list()
   for i in range(n): # enumerate all elements in a
      while len(b) > 0 and b[-1] >= a[i]: # compare the last element in b with a[i]
         b.pop() # remove the last element with O(1) complexity
      b.append(a[i]) # append a[i] to the end of b with O(1) complexity
   return b
```

The time complexity is $\mathcal{O}(n)$: each element in a is pushed to b once, and each element in b is popped at most once. It's important to remember that the complexity might not be tied to the number of nested loops.