CS 784: Computational Linguistics
Lecture 2.2: Probability and Information Theory

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[Some slides adapted from Madhur Tulsiani and David McAllester.]

Probability Spaces

Let Ω be a finite set. Let $P : \Omega \to [0, 1]$ be a function such that

$$\sum_{\omega \in \Omega} P(\omega) = 1.$$

We often refer to Ω as a **sample space** or **outcome space** and the function P as a **probability distribution** on this space.

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We often refer to Ω as a **sample space** or **outcome space** and the function P as a **probability distribution** on this space.

An **event** can be thought of as a subset of all possible outcomes, i.e., any $E \subseteq \Omega$ defines an event, and we define its probability as

$$\mathbb{P}[E] = \sum_{\omega \in E} P(\omega).$$

Random Variable and Expectation (Simplified)

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$$\mathbb{E}[X] = \sum_{\omega \in \Omega} P(\omega) X(\omega).$$

A random variable X is technically **neither random nor a variable**. However, we may informally understand it as a variable whose value is randomly drawn.

Probability Space and Random Variables: Example

Rolling a fair dice gives

$$\text{Sample space } \Omega = \{\text{one}, \text{two}, \text{three}, \text{four}, \text{five}, \text{six}\}$$

$$\text{Random variable } X : \Omega \to \mathbb{R} \text{ such that } X(\text{one}) = 1, \quad X(\text{two}) = 2, \ldots, X(\text{six}) = 6$$

$$\text{Probability } P(\omega) = \frac{1}{6} \qquad \forall \omega \in \Omega$$

$$\text{Event Prob. } \mathbb{P}[\text{even number}] = P(\text{two}) + P(\text{four}) + P(\text{six}) = \frac{1}{2}$$

$$\text{Event Prob. } \mathbb{P}[\text{number} \leq \text{two}] = P(\text{one}) + P(\text{two}) = \frac{1}{3}$$

$$\text{Expectation } \mathbb{E}[X] = \sum_{\omega \in \Omega} P(\omega) X(\omega) = 3.5$$

Probability Space and Random Variables: Example

If you'd like to change $X(\text{one}) = 6$, it also works:

$$\text{Sample space } \Omega = \{\text{one}, \text{two}, \text{three}, \text{four}, \text{five}, \text{six}\}$$

$$\text{Random variable } X : \Omega \to \mathbb{R} \text{ such that } X(\text{one}) = \mathbf{6},$$

$$X(\text{two}) = 2, \ldots, X(\text{six}) = 6$$

$$\text{Probability } P(\omega) = \frac{1}{6} \qquad \forall \omega \in \Omega$$

$$\text{Event Prob. } \mathbb{P}[\text{even number}] = P(\text{two}) + P(\text{four}) + P(\text{six}) = \frac{1}{2}$$

$$\text{Event Prob. } \mathbb{P}[\text{number} \leq \text{two}] = P(\text{one}) + P(\text{two}) = \frac{1}{3}$$

$$\text{Event Prob. } \mathbb{P}[X(\text{number}) \leq 2] = P(\text{two}) = \frac{1}{6}$$

$$\text{Expectation } \mathbb{E}[X] = \sum_{\omega \in \Omega} P(\omega) X(\omega) = \mathbf{4.33}$$

Conditional Probability

Conditioning on an event E is equivalent to restricting the probability space to E. The probability measure is then

$$P_E(\omega) = \begin{cases} \frac{P(\omega)}{\mathbb{P}[E]} & \forall \omega \in E, \\ 0 & \text{otherwise.} \end{cases}$$

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We define the **conditional probability** of an event F given E as

$$\mathbb{P}[F \mid E] = \sum_{\omega \in F} P_E(\omega) = \sum_{\omega \in E \cap F} \frac{P(\omega)}{\mathbb{P}[E]} = \frac{\mathbb{P}[E \wedge F]}{\mathbb{P}[E]}.$$

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We can calculate the conditional expectation similarly:

$$\mathbb{E}[X \mid E] = \sum_{\omega \in E} P_E(\omega) X(\omega).$$

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From the previous slide, we know that

$$\begin{aligned} \mathbb{P}[E \wedge F] &= \mathbb{P}[F \mid E]\mathbb{P}[E] \\ &= \mathbb{P}[E \mid F]\mathbb{P}[F]. \end{aligned}$$

Independence

Two non-zero probability events E and F are **independent** if $\mathbb{P}[E \mid F] = \mathbb{P}[E]$ (or $\mathbb{P}[F \mid E] = \mathbb{P}[F]$).

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Two random variables X and Y defined on the same finite probability space are independent if

$$\mathbb{P}[X = x \mid Y = y] = \mathbb{P}[X = x]$$

for all non-zero probability events $\{X = x\} := \{\omega : X(\omega) = x\}$ and $\{Y = y\} := \{\omega : Y(\omega) = y\}$.

Common Notations on Random Variables

In literature, $P(X = x)$ usually denotes “the probability that the random variable X takes on the value x”—this is, actually, $\mathbb{P}(\{\omega : X(\omega) = x\})$.

$\{\omega : X(\omega) = x\}$ is an event, with event probability applicable.

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- **Conditional probability** $P(X = x \mid Y = y) = P(x \mid y)$.
- **Joint probability** $P(X = x, Y = y) = P(x, y)$.
- **Marginal probability** $P(x) = \sum_y P(x, y)$, $P(y) = \sum_x P(x, y)$.
  X and Y are independent iff. $P(x, y) = P(x)P(y)$.
- **Expectation** $\mathbb{E}[X] = \mathbb{E}_{x\sim P}[x] = \sum_x x \cdot P(X = x) = \sum_x xP(x)$.

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In what follows, we will use the (intuitive) notations on this slide.

Why Information Theory?

Information theory arises in many places and many forms in computational linguistics and deep learning.

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Many model training objectives are derived from information theory.

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$H(X)$ is measured in bits if the base of the logarithm is 2.

It can be measured in nats if the base of the logarithm is e.

Shannon's Source Coding Theorem

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A prefix-free code for $\mathcal{X}$ assigns a bit (0/1) string $c(x)$ to each $x \in \mathcal{X}$ such that no $c(x)$ is a prefix of another $c(x')$.

- This ensures that the code, i.e., concatenation of arbitrarily many $c(x)$, is uniquely decodable.

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Theorem 1: For any c, we have $\mathbb{E}_{x \sim P}[|c(x)|] \geq H_2(X)$.

- See [this url] for a proof by Michael Langer.

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Theorem 2: There exists a prefix-free code c such that $\mathbb{E}_{x\sim P}[|c(x)|] \leq H_2(X) + 1$, by assigning each x a bit string of length $\lceil -\log_2 P(x) \rceil$.

Intuitive Example for the Source Coding Theorem

We have a random variable X that takes values from $\{a, b, c, d\}$ with probabilities $\{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{8}\}$.

There are two prefix-free codes c_1 and c_2:

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Expected encoding length:

$$2 \times \frac{1}{2} + 2 \times \frac{1}{4} + \\ 2 \times \frac{1}{8} + 2 \times \frac{1}{8} = 2$$

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- $c_2(a) = 0$
- $c_2(b) = 10$
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Expected encoding length:

$$1 \times \frac{1}{2} + 2 \times \frac{1}{4} + 3 \times \frac{1}{8} + 3 \times \frac{1}{8} = 1.75$$

Joint Entropy

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The joint entropy is a measure of the uncertainty in the joint distribution of X and Y.

Conditional Entropy

The **conditional entropy** of Y given X is defined as

$$
\begin{aligned}
H(Y \mid X) &= \sum_x P(x) H(Y \mid X = x) \\
&= -\sum_{x,y} P(x) P(y \mid x) \log P(y \mid x) \\
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Exercise: show that

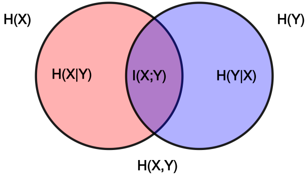
$$H(X, Y) = H(X) + H(Y \mid X) = H(Y) + H(X \mid Y).$$

Hint: expand everything.

Mutual Information

The **mutual information** between two random variables X and Y is defined as

$$\begin{aligned} I(X;Y) &= H(X) + H(Y) - H(X,Y) \\ &= H(X) - H(X \mid Y) \\ &= H(Y) - H(Y \mid X). \end{aligned}$$



[Figure from Wikipedia]

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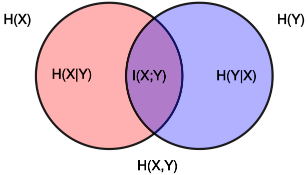

[Figure from Wikipedia]

It measures the amount of information that X and Y share: how much knowing one variable reduces uncertainty about the other.

Entropy and Cross Entropy

The entropy $H(X)$ of a random variable is the optimal (minimal) expected number of bits needed to encode a sample $x \sim P(x)$.

Can be also viewed as the **entropy of the distribution** P, $H(P)$.

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Let P and Q be two probability distributions over the same set. The cross entropy of P and Q is defined as

$$H(P, Q) = \mathbb{E}_{x \sim P}[-\log Q(x)]$$

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Plain language: the expected number of bits per sample is $H(P, Q)$, if we use the optimal code for Q to encode samples from P.

We will show $H(P, Q) \geq H(P)$.

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Not to be confused with the (one-distribution) joint entropy $H(X, Y)$!

The Kullback–Leibler Divergence

The Kullback–Leibler (KL) divergence of P from Q is defined as

$$D_{KL}(P \parallel Q) = \mathbb{E}_{x \sim P}\left[\log \frac{P(x)}{Q(x)}\right].$$

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The Kullback–Leibler (KL) divergence of P from Q is defined as

$$D_{KL}(P \parallel Q) = \mathbb{E}_{x \sim P}\left[\log \frac{P(x)}{Q(x)}\right].$$

Now we will prove that $D_{KL}(P \parallel Q) \geq 0$ for any P and Q.

Proof of $D_{KL}(P \parallel Q) \geq 0$: Jensen's Inequality

Definition (Convex Function)

A function $f: \mathbb{R} \to \mathbb{R}$ is convex if for all $x, y \in \mathbb{R}$ and $\lambda \in [0, 1]$,

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y).$$

Informally, a convex function is a function that upcurves everywhere.

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A convex function f satisfies Jensen's inequality:

$$f(\mathbb{E}[X]) \leq \mathbb{E}[f(X)]$$

Can be proved by induction on the number of samples.

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Can be proved by induction on the number of samples.

Example: $-\log(x)$ is a convex function.

Proving $D_{KL}(P \parallel Q) \geq 0$

$$
\begin{aligned}
D_{KL}(P \parallel Q) = \mathbb{E}_{x \sim P}\left[\log \frac{P(x)}{Q(x)}\right] &= \mathbb{E}_{x \sim P}\left[-\log \frac{Q(x)}{P(x)}\right] \\
&\geq -\log \mathbb{E}_{x \sim P}\left[\frac{Q(x)}{P(x)}\right] \\
&= -\log\left[\sum_x P(x) \frac{Q(x)}{P(x)}\right] \\
&= -\log\left[\sum_x Q(x)\right] \\
&= -\log 1 = 0
\end{aligned}
$$

Proving $D_{KL}(P \| Q) \geq 0$

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&= -\log\left[\sum_x Q(x)\right] \\
&= -\log 1 = 0
\end{aligned}
$$

Exercise: show that $I(X;Y) \geq 0$.

Hint: Express $I(X;Y)$ in the form of D_{KL}.

Cross Entropy and Kullback–Leibler Divergence

$$
\begin{aligned}
D_{KL}(P \parallel Q) &= \mathbb{E}_{x \sim P}\left[\log \frac{P(x)}{Q(x)}\right] \\
&= \mathbb{E}_{x \sim P}\left[\log P(x) - \log Q(x)\right] \\
&= H(P, Q) - H(P)
\end{aligned}
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About $H(P, Q)$: $D_{KL}(P \parallel Q) \geq 0 \Rightarrow H(P, Q) \geq H(P)$.

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About $H(P, Q)$: $D_{KL}(P \parallel Q) \geq 0 \Rightarrow H(P, Q) \geq H(P)$.
Suboptimal code for P will result in a longer expected code length.

Cross Entropy and Kullback–Leibler Divergence

$$
\begin{aligned}
D_{KL}(P \parallel Q) &= \mathbb{E}_{x\sim P}\left[\log \frac{P(x)}{Q(x)}\right] \\
&= \mathbb{E}_{x\sim P}\left[\log P(x) - \log Q(x)\right] \\
&= H(P, Q) - H(P)
\end{aligned}
$$

About $H(P, Q)$: $D_{KL}(P \parallel Q) \geq 0 \Rightarrow H(P, Q) \geq H(P)$.
Suboptimal code for P will result in a longer expected code length.

About $D_{KL}(P \parallel Q)$: it measures the inefficiency of using code for Q to encode samples from P, i.e., how many extra bits per sample are expected to be used.

Next

Statistical Methods